

$$\ln(ab) = \int_1^{ab} \frac{1}{t} dt$$

$$\int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt$$

$$t' = \frac{t}{a}$$

$$\frac{dt'}{dt} = \frac{1}{a} = \frac{t'}{t}$$

$$\int_a^{ab} \frac{1}{t} dt = \int_1^b \frac{1}{t'} dt'$$

$$\int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_1^b \frac{1}{t} dt$$

$$\ln(ab) = \ln(a) + \ln(b)$$

QED